TR/08/87 August 1987

A class of cubic and quintic spline modified collocation methods for the solution of two-point boundary value problems.

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Summary.

This paper is concerned with the study of a class of methods for solving second and fourth-order two-point boundary-value problems. The methods under consideration are modifications of the standard cubic and quintic spline collocation techniques, and are derived by making use of recent results concerning the a posteriori correction of cubic and quintic interpolating splines.

1. Introduction.

This paper is concerned with the study of a class of methods for computing smooth cubic and quintic spline approximations to solutions of second and fourth-order boundary-value problems for ordinary differential equations. This study has been motivated by the work of Daniel and Swartz [6] who proposed and analyzed a $0(h^4)$) cubic spline collocation scheme, the so-called extrapolated collocation method, for the solution of second-order problems. Here, we consider the problem of deriving a class of similar collocation schemes of high order, by making use of the results of Lucas [9] and our recent results [11], concerning the a posteriori correction of cubic and quintic interpolator splines.

Our specific objectives are as follows:

(i) To extend the work of [6] to a wider class of methods by making full use of the results of [9, 11], which were not available to the authors of [6].

(ii) To present a unified convergence analysis based on that of the extrapolated collocation method of [6], but covering the wider class of methods.

(iii) To show that the results of [9, 11] can also be used for computing derivative approximations of further increased accuracy, at any point of the interval under consideration.

The following notation will be used throughout the paper:

(i) π_n will denote a uniform partition,

$$\pi_n: \quad x_i = a + ih, \quad h = (b-a)/n; \quad i = 0,1.....n, \qquad (1.1)$$

of a bounded interval [a,b].

(ii) $S_m(\pi_n)$, $m \geq 1$, will denote the space of all smooth splines on [a,b], of degree m and with equally spaced knots (1.1), i.e.

$$S_m(\pi_n) := \left\{ s: \quad s \in c^{m-1}[a,b],, \text{ and on each of the subintervals of } \pi_n \text{ s is a polynomial of degree at most m.} \right\} \qquad (1.2)$$

(iii) $||.||$ will denote either the function norm $||.||_{L_\infty[a,b]}$ or the infinity vector and matrix norms.

(iv) $y_i^{(j)}$, $s_i^{(j)}$, e.t.c. will denotes $y^{(j)}(x_i)$, $s^{(j)}(x_i)$, e.t.c

2. Preliminary results.

The purpose of this section is to summarize the results of [11], on which much of the subsequent analysis is based. These results concern the a posteriori correction of non-periodic cubic and quintic splines, and are based on earlier results for odd degree periodic splines due to Lucas [9].

With r = 2 or r = 3, let s be a $S_{2r-1}(\pi_n)$-interpolate of a sufficiently smooth function y defined on [a,b]. That is, s is either a cubic (r = 2) or quintic (r = 3) spline, satisfying the n + 1 interpolation conditions $s_i = y_i$; = 0,1,...,n, and an appropriate set of 2r - 2 end conditions. Also, let $Y_M^{(j)}$ denote the corrected spline approximation to $y^{(j)}$, obtained as indicated in [11: Theor.2.2] by adding M $(1 \le M \le 3)$ correction terms to $s^{(j)}$. That is, for $1 \le M \le 3$, $0 \le \mu \le 1$ and $0 \le j \le 2r$,

$$Y_M^{(j)}(x_i + \mu h) = s^{(j)}(x_i + \mu h) + \sum_{m=0}^{M-1} \left\{ \frac{h^{2r-j+m}}{(2r+m)!} d_{i,M}^{(2r+m)} P_m^{(j)}(\mu) \right\}; \; i = 0,1,\dots,n-1, \quad (2.1)$$

where P_m; m = 0,1,2, are the polynomials listed in (2.2)-(2.3) below, and $d_{i,M}^{(2r+m)}$ are spline approximations to the derivatives $y_i^{(2r+m)}$ of y. These approximations are given by linear combinations of the spline derivatives $s_i^{(2r-2)}$ as indicated in (2.4)-(2.5) below.

The polynomials P_m ; m = 0,1,2, in (2.1) are as follows:

(i) When r = 2, i.e. when s is a cubic spline,

$$P_0(\mu) = \mu^4 - 2\mu^3 + \mu^2, \qquad P_1(\mu) = \mu^5 - \frac{5}{3}\mu^3 + \frac{2}{3}\mu, \qquad P_2(\mu) = \mu^6 - \mu^2. \qquad (2.2)$$

(ii) When r = 3, i.e. when s is a quintic spline,

$$\left.\begin{array}{l} P_0(\mu) = \mu^6 - 3\mu^5 + \frac{5}{2}\mu^4 - \frac{1}{2}\mu^2, \qquad P_1(\mu) = \mu^7 - \frac{7}{2}\mu^5 + \frac{7}{2}\mu^3 - \mu \\ \qquad P_2(\mu) = \mu^8 - 7\mu^4 + 6\mu^2. \end{array}\right\} \qquad (2.3)$$

□

Let

$$\tilde{y}_i^{(2r)} := \frac{1}{h^2}\left\{s_{i-1}^{(2r-2)} - 2s_i^{(2r-2} + s_{i+1}^{(2r-2}\right\}; \quad i = 1,2,\ldots n-1. \tag{2.4}$$

Then, the values $d_{i,M}^{(2r)}$ in (2, 1) are as follows:

(1)
$$d_{i,1}^{(2r)} = d_{i,2}^{(2r)} = d_{i,3}^{(2r)} = \tilde{y}_i^{(2r)}; \quad i = 1,2,\ldots,n-1, \tag{2.5a}$$

and

$$d_{0,1}^{(2r)} = \tilde{y}_1^{(2r)},\ d_{0,2}^{(2r)} = 2\tilde{y}_1^{(2r)} - \tilde{y}_2^{(2r)},\ d_{0,3}^{(2r)} = 3\tilde{y}_1^{(2r)} - 3\tilde{y}_2^{(2r)} + \tilde{y}_3^{(2r)}. \tag{2.5b}$$

The remaining values $d_{i,M}^{(2r+1)}$, $2 \le M \le 3$, and $d_{i,3}^{(2r+2)}$ are given in terms of the expressions (2.5a, b) and the three additional expressions

$$d_{n,1}^{(2r)} = \tilde{y}_{n-1}^{(2r)},\ d_{n,2}^{(2r)} = 2\tilde{y}_{n-1}^{(2r)} - \tilde{y}_{n-2}^{(2r)},\ d_{n,3}^{(2r)} = 3\tilde{y}_{n-1}^{(2r)} - 3\tilde{y}_{n-2}^{(2r)} + \tilde{y}_{n-3}^{(2r)}, \tag{2.5c}$$

as indicated in (ii), (iii) below:

(ii)
$$d_{i,M}^{(2r+1)} = \frac{1}{2h}\left\{d_{i+1,M}^{(2r)} - d_{i-1,M}^{(2r)}\right\},\ 2 \le M \le 3;\ i = 1,2,\ldots n-1, \tag{2.6a}$$

and

$$d_{0,2}^{(2r+1)} = d_{1,2}^{(2r+1)},\ d_{0,3}^{(2r+1)} = 2d_{1,3}^{(2r+1)} - d_{2,3}^{(2r+1)}. \tag{2.6b}$$

(iii)
$$d_{i,3}^{(2r+2)} = \frac{1}{h^2}\left\{d_{i-1,3}^{(2r)} - 2d_{i,3}^{(2r)} + d_{i+1,3}^{(2r)}\right\}; \quad i = 1,2,\ldots,n-1, \tag{2.7a}$$

and

$$d_{0,3}^{(2r+2)} = d_{1,3}^{(2r+2)} \quad . \tag{2.7b}$$

□

Remark 2.1 As was previously remarked the values $d_{i,M}^{(2r+m)}$, given by (2.4)-(2.7), are approximations to the derivatives y. $y_i^{(2r+m)}$, of y. For the order of these approximations see Remark 2.4 below.

□

Remark 2.2 It is important to observe that for i = 1,2,... ,n-1, the values $d_{i,M}^{(2r)}$ in (2.1) are independent of the number M of correction terms used i.e., from (2.5a),

$$d_{i,1}^{(2r)} = d_{i,2}^{(2r)} = d_{i,3}^{(2r)} = \frac{1}{h^2}\left\{s_{i-1}^{(2r-2)} - 2s_{i+1}^{(2r-2)} + s_{i+1}^{(2r-2)}\right\}, \quad i = 1,2,\ldots,n-1. \tag{2.8}$$

Similarly, from (2.6a) and (2.7a),

$$d_{i,2}^{(2r+1)} = d_{i,3}^{(2r+1)} = \frac{1}{2h^3}\left\{-s_{i-2}^{(2r-2)} + 2s_{i-1}^{(2r-2)} - 2s_{i+1}^{(2r-2)} + s_{i+2}^{(2r-2)}\right\};$$

$$i = 2,3,\ldots,n- \qquad (2.9)$$

and

$$d_{i,3}^{(2r+2)} = \frac{1}{h^4}\left\{s_{i-2}^{(2r-2)} - 4s_{i-1}^{(2r-2)} + 6s_{i}^{(2r-2)} - 4s_{i+1}^{(2r-2)} + s_{i+2}^{(2r-2)}\right\};$$

$$i = 2,3,\ldots,n-2. \qquad (2.10)$$

Therefore, the subscript M is needed only because the values $d_{0,M}^{(2r)}$ and $d_{i,M}^{(2r+1)}$; i = 0,1, n-1, which approximate respectively the "end" derivatives $y_0^{(2r)}$ and $y_i^{(2r+1)}$; i = 0,1,n-1, depend on the number of correction terms used.

□

We can now state the main result concerning the corrected spline approximations $Y_M^{(j)}$ as follows; see [11: Theor.2.2] and also [9: Theor.4].

Theorem 2.1. Let $Y_0^{(j)} := s^{(j)}$ and let $Y_M^{(j)}$, $1 \le M \le 3$, be the corrected spline approximations defined by equations (2.1)-(2.7). Then, for $0 \le M \le 3$, $0 \le \mu \le 1$ and $0 \le j \le 2r$,

$$y^{(j)}(x_i + \mu h) = Y_M^{(j)}(x_i + \mu h) + 0(h^{2r-j+M}); \quad i = 0,1,\ldots,n-1, \qquad (2.11)$$

provided that $y \in C^{2r+M}[a,b]$, and the end conditions of s are of order $p \ge 2+M$ in the sense of Definition 2.1 of [11: p.491].

□

We end this section by making several remarks concerning the correction formula (2.1) and the result (2.11) of Theorem 2.1.

Remark 2.3. The assumption concerning the order p of the end conditions of the spline s is necessary for Theorem 2.1 to hold and, in this sense, it is also necessary for the analysis of the collocation methods considered in the present paper. However, as will become apparent later, the actual end conditions of s do not play an explicit role in the analysis or the implementation of the methods. For this reason, there is no need for us to repeat here the criteria used in [for determining the order of end conditions.

□

Remark 2.4. Let $y \in c^{2r+M}$ [a,b], $1 \leq M \leq 3$, and let the end conditions of s be of order $p \geq 2 + M$. Then, for $i = 0,1,\ldots,n-1$, the values $d_{i,M}^{(2r+m)}$ in (2.1) satisfy the following:

$$y_i^{(2r)} = d_{i,M}^{(2r)} + 0(h^M), \qquad 1 \leq M \leq 3, \tag{2.12}$$

$$y_i^{(2r+1)} = d_{i,M}^{(2r+1)} + 0(h^{M-1}), \qquad 2 \leq M \leq 3, \tag{2.13}$$

and

$$y_i^{(2r+2)} = d_{i,M}^{(2r+2)} + 0(h), \qquad M = 3; \tag{2.14}$$

see [11: p.p. 492-93] and also [9: Theor.3].

□

Remark 2.5. With M = 0, (2.11) shows that if $y \in c^{2r}[a,b]$ and the end conditions of s are of order $p \geq 2$, then

$$\| s^{(j)} - y^{(j)} \| = 0(h^{2r-j}), 0 \leq j \leq 2r-1 \| \tag{2.15}$$

In addition, (2.11) gives the points in $[x_i, x_{i+1}]$ where the derivatives of s display superconvergence. Thus, if $y \in c^{2r+1}[a,b]$ and the end conditions of s are of order $p \geq 3$, then it follows from (2.11) that

$$y^{(j)}(x_i + \mu_j h) = s^{(j)}(x_i + \mu_j h) + 0(h^{2r-j+1}), \quad 0 \leq j \leq 2r-1; \quad i = 0,1,\ldots,n-1, \tag{2.16}$$

where the μ_j denote respectively the zeros of the polymomials $p_0^{(j)}; j = 1,2,\ldots,2r-1$, in [0,1] These zeros are as follows:

(i) If r=2, i.e. if s is a cubic spline, then

$$\mu_1 = 0, \tfrac{1}{2}, 1, \quad \mu_2 = \tfrac{1}{2} \pm \tfrac{1}{6}\sqrt{3}, \mu_3 = \tfrac{1}{2}. \tag{2.17}$$

(ii) If r=3, i.e. if s is a quintic spline, then

$$\left.\begin{array}{c} \mu_1 = 0, \frac{1}{2}, 1 \mu_1 = \frac{1}{2} \pm \sqrt{(\frac{1}{4} - \frac{1}{30}\sqrt{30})}, \mu_3 = 0, \frac{1}{2}, 1 \\ \mu_4 = \frac{1}{2} \pm \frac{1}{6}\sqrt{3}, \ \mu_5 = \frac{1}{2}. \end{array}\right\} \tag{2.18}$$

We note in particular that the odd derivatives $s^{(2j+1)}, 0 \leq j \leq r-2$, display superconvergence at the knots, i.e.

$$y_i^{(2j+1)} = s_i^{2j+1} + 0(h^{2r-2j}), \ 0 \leq j \leq r-2; \quad i = 0,1,\ldots,n, \tag{2.19}$$

whilst the best order of convergence which can be achieved by the even derivatives at the knots is that given by (2.15), i.e.

$$y_i^{(2j)} = s_i^{(2j)} + 0(h^{2r-2j}), \quad 1 \le j \le r; i = 0,1,.....,n-1. \tag{2.20}$$

□

<u>Remark 2.6</u> In both the cubic (r = 2) and quintic (r=3) cases, the first r-1 odd derivatives of P_0 (μ), the first r even derivatives of $P_1(\mu)$ and the first r odd derivatives of $P_2(\mu)$ are zero when $\mu = 0$. Thus, Theorem 2.1 implies the following:

(i) If y Є C^{2r+2} [a,b] and the end conditions of s are of order p≥4, then for 1≤j≤r,

$$y_i^{(2j)} = s_i^{(2j)} + \frac{h^{2r-2j}}{(2r)!} p_0^{(2j)}(0) d_{i,2}^{(2r)} + 0(h^{2r-2j+2}); i = 0,1,....,n-1, \tag{2.21a}$$

i.e.,

$$y_i^{(2j)} = Y_{1*}^{(2j)}(x_i) + 0(h^{2r-2j+2}); i = 0,1,....,n-1, \tag{2.21b}$$

where y_{1*} denotes the corrected spline approximation Y_1 but with the value $d_{0,1}^{(2r)}$ replaced by $d_{0,2}^{(2r)}$; see Remark 2.2.

(ii) If y Є C^{2r+3} [a,b] and the end conditions of s are of order p ≥ 5, then for 1 ≤ j ≤ r -1,

$$y_i^{(2j-1)} = s_i^{(2j-1)} + \frac{h^{2r-2j+2}}{(2r+1)!} p_1^{(2j-1)}(0) d_{i,3}^{(2r+1)} + 0(h^{2r-2j+4}); \tag{2.22a}$$

i = 0,1,….,n-1,

i.e.,

$$y_i^{(2j-1)} = y_{2*}^{(2j-1)}(x_i) + 0(h^{2r-2j+4}); \quad i = 0,1,....,n-1$$

where y_{2*} denotes the corrected spline approximation Y_2 but with the values $d_{i,2}^{(2r+1)}$; i = 0,1,n-1, replaced respectively by $d_{i,3}^{(2r+1)}$; i=0,1,n-1.

□

<u>Remark 2.7.</u> If $y \in c^{2r+4}$ [a,b] and the end conditions of s are of order p ≥ 6, then it can be shown that for 1 ≤ j ≤ r,

$$y_i^{(2j)} = y_3^{(2j)}(x_i) + 0(h^{k2j}); i = 2,3,....,n-1, \tag{2.23}$$

where $k_{2j} = 2r - 2j + 4$ rather than $k_{2j} = 2r - 2j + 3$ as predicted by Theorem 2.1;

see [11: Remark 2.3 (ii)]. Since

$$y_3^{(2j)}(x_i) = s_i^{(2j)} + \frac{h^{2r-2j}}{(2r)!} p_0^{(2j)}(0) d_{i,3}^{(2r)} + \frac{h^{2r-2j+2}}{(2r+2)!} p_2^{(2j)}(\theta) d_{i,3}^{(2r+2)}, \tag{2.24}$$

it follows easily from (2.23) that for $1 \le j \le r$,

$$y_i^{(2j)} = y_3^{(2j)}(x_i) + 0(h^{2r-2j+4}); i = 0,1,....,n-1, \tag{2.25}$$

where Y_{3*} denotes the corrected approximation Y_3, given by (2.1)-(2.7), but with the expressions $d_{0,3}^{(2r)}, d_{n,3}^{(2r)}$ and $d_{0,3}^{(2r+2)}$ in (2.5b), (2.5c) and (2.7b) replaced respectively by the following more "accurate" expressions:

$$d_{0\ 3}^{(2r)} = 4\tilde{y}_1^{(2r)} - 6\tilde{y}_2^{(2r)} + 4\tilde{y}_3^{(2r)} - \tilde{y}_4^{(2r)}, \tag{2.26}$$

$$d_{n,3}^{(2r)} = 4\tilde{y}_{n-1}^{(2r)} - 6\tilde{y}_{n-21}^{(2r)} + 4\tilde{y}_{n-3}^{(2r)} - \tilde{y}_{n-4}^{(2r)}, \tag{2.27}$$

$$d_{0,3}^{(2r+2)} = 2d_{1,3}^{(2r+2)} - d_{2,3}^{(2r+2)}; \tag{2.28}$$

see the discussion in p.494 of [11].
□

Remark 2.8. Because the values $d_{i,M}^{(2r+m)}$ are given by the expressions (2.4)-(2.7), the "corrections" under the summation sign in (2.1) are in terms of the spline derivatives $s_i^{(2r-2)}$; i.e. in terms of the second derivatives $s_i^{(2)}$ in the cubic case, and the fourth derivatives $s_i^{(4)}$ in the quintic case. However, it follows easily from the analysis of [9] and [11] that the corrections can also be expressed in terms of other spline derivative values. This can be done by replacing the expressions (2.4) for $\tilde{y}_i^{(2r)}$, and (2.5) for $d_{i,M}^{(2r)}$, by other suitable approximations to $y_i^{(2r)}$. The essential requirement for such alternative representations is that the new approximations $d_{i,M}^{(2r)}$ also satisfy (2.12), under the hypotheses of Remark 2.4. For example, in the quintic (r = 3) case the two corrections in Y_2 can be expressed in terms of second derivatives, by replacing (2.4) and the formulae for $d_{i,2}^{(6)}$ in (2.5) respectively by :

$$y_i^{(6)} = \frac{1}{h^4}\left\{s_{i-2}^{(2)} - 4s_{i-1}^{(2)} + 6s_i^{(2)} - 4s_{i+1}^{(2)} + s_{i+2}^{(2)}\right\}; i = 2,3,....,n-2 \tag{2.29}$$

$$d_{i,2}^{(6)} = \tilde{y}_i^{(6)}; i = 2,3,\ldots,n-2, \tag{2.30a}$$

$$d_{o,2}^{(6)} = 3\tilde{y}_2^{(6)} - 2\tilde{y}_3^{(6)}, d_{o,2}^{(6)} = 2\tilde{y}_2^{(6)} - \tilde{y}_3^{(6)}, \tag{2.30b}$$

and

$$d_{n-1,2}^{(6)} = 2\tilde{y}_{n-2}^{(6)} - \tilde{y}_{n-3}^{(6)}, d_{n,2}^{(6)} = 3\tilde{y}_{n-2}^{(6)} - \tilde{y}_{n-3}^{(6)}. \tag{2.30c}$$

3. A class of modified collocation methods for linear problems.

In this section we describe in general terms a class of collocation-type methods for the solution of second and fourth-order two-point boundary-value problems of the form:

$$[y] := y^{(2r-2)}(x) + \sum_{j=0}^{2r-3} e_j(x) y^{(j)}(x) = f(x), \quad x \in [a,b] \tag{3.1a}$$

$$By = 0, \tag{3.1b}$$

where either r = 2 or r = 3, and where (3.1b) denotes a set of 2r - 2 linearly independent boundary conditions of the form

$$\sum_{j=0}^{2r-3} \left\{\alpha_{ij} y^{(j)}(a) + \beta_{ij} y^{(j)}(b)\right\} = 0, \quad r = 2,3, \quad 0 \le i \le 2r-3. \tag{3.1b}$$

Our main purpose is to provide a motivation for such methods, and to set up suitable notation for use in subsequent sections.

Let π_n be the uniform partition (1.1) of the interval [a.b], and let s be a $s_{2r-1}(\pi_n)$-interpolate of the solution y of (3.1). (That is, s is either a cubic or a quintic spline depending on whether (3.1) is a second-order (r =2) or fourth-order (r =3) problem.) Also, let y and s satisfy the smoothness and continuity requirements of Remark 2.5 for the result (2.15) to hold. Then, it follows from the equations

$$[y](x_i) := y_i^{(2r-2)} + \sum_{j=0}^{2r-3} e_j(x_i) y_i^{(j)} = f_i + 0(h^2); i = 0,1,\ldots,n, \tag{3.2}$$

and the boundary conditions (3.1b) that

$$s_i^{(2r-2)} + \sum_{j-0}^{2r-3} e_j(x_i) s_i^{(j)} = f_i + 0(h^2); i = 0,1,\ldots,n, \tag{3.3}$$

and

$$\mathcal{B}s = 0(h^k),\ k \geq 3; \tag{3.4}$$

see Remark 2.5. The above two results lead naturally to the well-known collocation method, where spline $\tilde{s}$ approximating the solution of (3.1) is obtained from (3.3) and (3.4) by simply dropping the $0(h^2)$ and $0(h^k\ )$ terms. That is, $\tilde{s}$ is defined by the n + 2r - 1 linear equations

$$\tilde{s}_i^{(2r-2)} + \sum_{j=0}^{2r-3} e_j(x_i)\tilde{s}_i^{(j)} = f_i; \qquad i = 0,1,\ldots,n, \tag{3.5a}$$

and

$$\tilde{s} = 0 \tag{3.5b}$$

The "extrapolated collocation method" of Daniel and Swartz [6] is similar to the above collocation method, but the approximating spline $\tilde{s}$ is defined by a different linear system. More precisely, the defining equations consist of the same 2r - 2 "boundary equations" (3.4), but in this case the n + 1 equations approximating the differential equation at the knots are obtained from (3.2) as follows:

(a) The values y_i and the derivatives $y_i^{(j)}, 1 \leq j \leq 2r-3,$ are replaced by $\tilde{s}_i$ and the corresponding spline derivatives $\tilde{s}_i^{(j)}, 1 \leq j \leq 2r-3,$ (That is the replacement of $y_i^{(j)}, 1 \leq j \leq 2r-3,$ is the same as in the collocation method.)

(b) The derivatives $y_i^{(2r-2)}; i = 0,1,\ldots,n,$ are replaced by linear combinations of the spline derivatives $\tilde{s}_i^{(2r-2)}$ as follows:

$$y_0^{(2r-2)} \to \tilde{s}_0^{(2r-2)} + \tfrac{1}{12}\left\{2\,\tilde{s}_0^{(2r-2)} - 5\,\tilde{s}_1^{(2r-2)} + 4\,\tilde{s}_2^{(2r-2)} - \tilde{s}_3^{(2r-2)}\right\}, \tag{3.6a}$$

$$y_i^{(2r-2)} \to \tilde{s}_0^{(2r-2)}\tfrac{1}{12}\left\{\tilde{s}_{i-1}^{(2r-2)} - 2\,\tilde{s}_i^{(2r-2)} + \tilde{s}_{i+1}^{(2r-2)}\right\}, \qquad i = 1,2,\ldots,\ n-1, \tag{3.6b}$$

$$y_n^{(2r-2)} \to \tilde{s}_n^{(2r-2)} + \tfrac{1}{12}\left\{-\,\tilde{s}_{n-3}^{(2r-2)} + 4\tilde{s}_{n-2}^{(2r-2)} - 5\tilde{s}_{n-1}^{(2r-2)} + 2\tilde{s}_n^{(2r-2)}\right\} \tag{3.6c}$$

<u>Remark 3.1.</u> The method proposed in [6] is for the case r = 2 only, and is based on the observation that the linear combinations (3.6) with r = 2 and with $\tilde{s}$ replaced by s , are $0(h^4)$ approximations to $y_i^{(2)}$. In other words, the paper of Daniel and Swartz is concerned only with the cubic spline solution of second order

problems. However, as will become apparent in Section 5.2, their method can be extended trivially to the case r=3, by making use of the a posteriori correction results of Section 2. □

Both the collocation and the extrapolated collocation methods may be regarded as special cases of a more general class of methods, in which the replacement of the derivatives $y_i^{(j)}$ in (3.2) is performed by using formulae of the type

$$y_i^{(j)} = s_i^{(j)} + L_i^{\{j\}}\left[s^{(2r-2)}\right] + 0\left(h^{\ell j}\right), \quad 1 \le j \le 2r-2; \quad i = 0,1,....,n, \tag{3.7}$$

where the notation $L_i^{\{j\}}[g]$ has the following meaning: "Given a function g defined on [a,b], $L_i^{\{j\}}[g]$ denotes a linear combination of values of g at a small number of points of the subdivision π_n, near the point x_i " For example, in the collocation method

$$L_i^{\{j\}}[g] = 0, \quad 1 \le j \le 2r-2; \quad i = 0,1,....,n, \tag{3.8}$$

whilst in the extrapolated collocation method of [6],

$$L_i^{\{j\}}[g] = 0, \quad 1 \le j \le 2r-3; \quad i = 0,1,....,n, \tag{3.9a}$$

$$L_0^{\{j\}}[g] = \tfrac{1}{12}\{2g_0 - 5g_1 + 4g_2 - g_3\}, \tag{3.9b}$$

$$L_i^{\{2r-2\}}[g] = \tfrac{1}{12}\{g_{i-1} - 2g_i + g_{i+1}\}; \quad i = 1,2,...,n-1, \tag{3.9c}$$

and

$$L_n^{\{2r\text{-}2}[g] = \frac{1}{12}\{-g_{n-3} + 4g_{n-2} - 5g_{n-1} + 2g_n\}. \tag{3.9d}$$

More generally, the corrected spline approximations, defined by (2.1)-(2.7), give

$$y_i^{(j)} = Y_M^{(j)}(x_i) + 0(h^{2r-j+M}); \qquad i = 0,1,...,n, \tag{3.10}$$

and these formulae are of the type (3.7),

The equations (3.7) can be expressed more compactly as

$$\underline{y}^{(j)} = \underline{s}^{(j)} + \Lambda_{n,j}\underline{s}^{(2r-2)} + \underline{0}(h^{\ell j}), \qquad 1 \le j \le 2r-2, \tag{3.11}$$

where $\underline{y}^{(j)}$ and $\underline{s}^{(j)}$ are the (n +1)-dimensional column vectors

$$\underline{y}^{(j)} = \left\{\underline{y}_i^{(j)}\right\}_{i=0}^{n} \quad \text{and} \quad \underline{s}^{(j)} = \left\{\underline{s}_i^{(j)}\right\}_{i=0}^{n}$$

is an (n + 1) x (n + 1) matrix whose form depends on the coefficients of the linear combinations $L_i^{\{j\}}[.]$, and $\underline{0}(h^{\ell_j})$ is an (n + 1)-dimensional column vector whose components are all $0(h^{\ell_j})$. With this notation, the substitution of the expressions (3.7) into the equations (3.2) gives an (n + 1) × (n + 1) linear system of the form

$$A_n \underline{s}^{(2r-2)} + \sum_{j=0}^{2r-3} \Delta_{n,j}\underline{s}^{(j)} = \underline{f} + \underline{0}(h^{\ell}), \qquad \ell = \min_j\{\ell_j\}, \tag{3.12a}$$

where:

(a) $\Delta_{n,j}, 0 \le j \le 2r-3$, are the (n + 1) × (n + 1) diagonal matrices

$$\Delta_{n,j} = \text{diag}\left\{e_j(x_0), e_j(x_1), \ldots . e_j(x_n)\right\}. \tag{3.12b}$$

(b) $\mathcal{A}_n$ is the (n +1) × (n + 1) matrix

$$\mathcal{A}_n = I + \Lambda_{n,2r-2} + \sum_{j=1}^{2r-3} \Delta_{n,j}\Lambda_{n,j}. \tag{3.12c}$$

(c) $\underline{f}$ is the (n + 1)-dimensional column vector

$$\underline{f} = \{f_i\}_{i=0}^{n}. \tag{3.12d}$$

In what follows, a method where the approximating spline $\tilde{s}$ is determined from (3.4) and a set of equations of the form (3.12), by dropping the $0(h^k)$ and $\underline{0}(h^{\ell})$ terms, will be referred to as a "modified collocation method". That is, in such a method the spline $\tilde{s}$ will be defined by a set of n + 2r-1 linear equations of the form

$$\mathcal{A}_n \underline{\tilde{s}}^{(2r-2)} + \sum_{j=0}^{2r-3} \Delta_{n,}\underline{\tilde{s}}^{(j)} = \underline{f}\,, \tag{3.13a}$$

and

$$\tilde{s} = 0. \tag{3.13b}$$

Remark 3.2. It is important to observe that in the class of methods defined by (3.13), the derivatives in the boundary conditions (3.1b) are always replaced by the corresponding derivatives of $\tilde{s}$.

□

Remark 3.3. In the standard collocation method, all the matrices $\Lambda_{n,j}$ in (3.12c) are null. Thus, (3.13a) simplifies to

$$\underline{\tilde{s}}^{(2r-2)} + \sum_{j=0}^{2r-3} \Delta_{n,j} \underline{\tilde{s}}^{(j)} = \underline{f}, \tag{3.14}$$

which is the matrix form of equations (3,5).

□

4. Convergence for linear problems.

This section is concerned with the method of analysis used by Daniel and Swartz [6: §4], for establishing the convergence of the extrapolated collocation method. Our purpose here is to show that the same method can be used, more generally, for the analysis of modified collocation methods defined by linear systems of the form (3.13).

We first make the following three assumptions concerning the boundary value problem (3.1):

A4.1. The functions $e_j, 0 \le j \le 2r-3$, and f in the differential equation (3.1a) are at least continuous on [a,b].

A4.2. The boundary value problem (3.1) has a unique solution $y \in C^m$ [a,b], where $m \ge 2r$.

A4.3. The equation $y^{(2r-2)} = 0$ with boundary conditions (3.1b) has only the trivial solution.

The above assumptions guarantee the existence of a Green's function G(x,t) associated with the differential operator D^{2r-2} and the boundary conditions (3.1b), so that if

$$v := y^{(2r-2)}, \tag{4.1}$$

then

$$y^{(j)}(x) = \int_a^b \frac{\partial^j G(x,t)}{\partial x^j} v(t)dt, \qquad 0 \le j \le 2r-3. \tag{4.2}$$

Similarly, if

$$v_n := \tilde{s}^{(2r-2)}, \tag{4.3}$$

where $\tilde{s}$ is the approximating spline defined by the linear system (3.13), then

$$\tilde{s}^{(j)}(x) = \int_a^b \frac{\partial^j G(x,t)}{\partial x^j}\; v_n\,(t)dt, \quad 0 \le j \le 2r-3. \tag{4.4}$$

Proceeding as in [6], we next introduce the following three operators:

(i) $$\mathcal{D}_n : C[a,b] \to IR_{n+1}, \tag{4.5a}$$

where for any $g \in C[a, b]$,

$$(\mathcal{D}_n g)_i = g(x_i) \qquad i = 0,1,\ldots.n. \tag{4.5b}$$

(ii) $$\mathcal{M}_n : IR_{n+1} \to S_1(\pi_n), \tag{4.6a}$$

via piecewise linear interpolation at the points $\{x_i\}_{i=0}^n$. That is, for any vector $\underline{z} \in IR_{n+1}$,

$$\mathcal{M}_n \underline{z} = \frac{1}{h}\,\{(x_{i+1}-x)z_i \;+\; (x-x_i)z_{i+1}\}\;,$$

$$x \in [x_i,\; x_{i+1}]; \qquad i = 0,1,\ldots n-1. \tag{4.6b}$$

(iii) $$K:\; C[a,b] \to\; C[a,b] \tag{4.7a}$$

where for any $g \in C[a,b]$,

$$(kg)\;(x) = \sum_{j=0}^{2r-3}\left\{e_j(x)\int_a^b \frac{\partial^j G(x,t)}{\partial x^j}\;\; g(t)\;\,dt\right\}\;. \tag{4.7b}$$

Then, solving the boundary value problem (3.1) for y is equivalent to solving for $v = y^{(2r-2)}$ in

$$(I+K)\;v = f. \tag{4.8}$$

Similarly, solving for $\tilde{s}$ in equations (3.13) is equivalent to solving for $v_n = \tilde{s}^{(2r-2)}$ in

$$A_n \mathcal{D}_n v_n + \mathcal{D}_n K v_n = \mathcal{D}_n f\;\;, \tag{4.9}$$

where A_n is the matrix (3.12c). We make the following two assumptions regarding this matrix:

<u>A4.4</u> A_n is uniformly bounded and there exists $n_0 > 0$ so that, for $n \ge n_0$, A_n possesses a uniformly bounded inverse A_n^{-1}.

A4.5 For each fixed $u \in C\ [a,b]$

$$\lim_{n\to\infty} \|D_n u - A_n D_n u\| = 0.$$

The operators D_n and M_n defined by (4.5) and (4.6) are respectively the restriction and prolongation operators involved in the analysis of Daniel and Swartz; see [6: Defs (4.1), (4.2)]. Also, when r=2 the operator K coincides with the corresponding operator in [6: Def. (4.3)]. Finally, because of the assumptions- A4:4 and A4.5, the matrix A_n can take the place of the matrix Q_n involved in [6]. Thus, the three results stated below can be deduced immediately from the analysis of [6: p.p.166-168].

(i) For $n \geq n_0$,, equation (4.9) can be written as

$$(I + P_n K)v_n = P_n f, \tag{4.10}$$

where

$$P_n := M_n A_n^{-1} D_n, \tag{4.11}$$

defines a sequence of operators converging strongly to the identity operator on C[a,b].

(ii) For sufficiently large n, $(I + P_n K)^{-1}$ exists and is uniformly bounded. Thus, equation (4.10), or equivalently equation (4.9), has a unique solution v_n.

(iii) The solution v_n of (4.10) converges uniformly to the solution v of (4.8). In other words, if the assumptions A4.1 -A4.5 hold then the modified collocation method corresponding to the equations (3.13) is well-defined, and the derivative $\tilde{s}^{(2r-2)}$ the resulting approximating spline $\tilde{s}$ converges uniformly to $y^{(2r-2)}$. Furthermore, by modifying in an obvious manner the analysis of [6; p. 168], it is easy to show that

$$\left\|s^{(j)} - \tilde{s}^{(j)}\right\| = 0(h^{\beta}), \quad 0 \leq j \leq 2r-2, \tag{4.12a}$$

with

$$= \min\ \{k, \ell\}, \tag{4.12b}$$

where s is the $S_{2r-1}(\pi_n)$-interpolate of y defined in Section 3, and k, ℓ are respectively the orders of approximation of the spline replacements (3.4) and (3.12). Hence,

$$\left\| \tilde{s}^{(j)} - y^{(j)} \right\| = 0(h^{\gamma}), \quad 0 \le j \le 2r-2, \tag{4.13a}$$

where

$$\gamma = \min\{2r - j, k, \ell\}. \tag{4.13b}$$

This last result is obtained from (4.12) by use of the triangle inequality, because the continuity assumption A4.2 implies that

$$\left\| s^{(j)} - y^{(j)} \right\| = 0(h^{2r-j}), \quad 0 \le j \le 2r-1. \tag{4.14}$$

5. Examples of modified collocation methods.

In this section we illustrate the use of the a posteriori correction results of Section 2 and of the analysis of Daniel and Swartz [6] outlined in Section 4, for deriving and analyzing modified collocation methods of the type described in Section 3. In particular, we show that the cubic spline extrapolated collocation method of [6] extends immediately to a $0(h^4)$ quintic spline method for fourth-order problems, and we also derive a new $0(h^G)$ method for such problems. In addition, we explain how the results of Section 2 can be used to provide derivative approximations of further increased accuracy at any point of the interval [a,b].

In the examples that follow we always assume that the conditions A4.1-A4.3 concerning the boundary value problem (3.1) are satisfied and, with reference to A4.2, we indicate the required continuity class of y. Then, in order to establish the convergence of a particular modified collocation method we need only show that the conditions A4.4 and A4.5 concerning the corresponding matrix A_n hold; see Section 4. That is, for convergence we need only prove that:

(i) A_n is uniformly bounded and, for $n \ge n_0$, it possesses a uniformly bounded inverse A_n^{-1}.

(ii) For each fixed $u \in C\,[a,b]$, $\lim_{n\to\infty} \|D_n u - A_n D_n u\| = 0$, where D_n is the restriction operator defined by (4.5).

5.1 Standard collocation.

Here we assume that $y \in C^{2r}[a,b]$, and that the end conditions of s are of order $p \ge 2$, so that

$$\left\| s^{(j)} - y^{(j)} \right\| = 0(h^{2r-j}), \quad 0 \le j \le 2r-1, \tag{5.1}$$

and in particular

$$y_i^{(2e-2)} = s_i^{(2r-2)} + 0(h^2); \qquad i = 0,1,\ldots,n. \tag{5.2}$$

Thus, the exponents k and ℓ in (4.13b) are respectively k $\geq$ 3 and ℓ = 2; see equations (3.3) and (3.4). Also, as we indicated in Section 3, in this case the matrix A_n is the identity matrix. Therefore, the conditions A4.4 and A4.5 are satisfied trivially, and (4.13) gives

$$\left\| \tilde{s}^{(j)} - y^{(j)} \right\| = 0(h^2), \qquad 0 \le j \le 2r-2, \tag{5.3}$$

which is one of the results established by Russell and Shampine [13].

5.2 The extrapolated collocation method of Daniel and Swartz [6].

Here we assume that $y \in C^{2r+2}$ [a,b] and that the end conditions of s are of order p $\geq$ 4. Then,

$$\left\| y^{(j)} - s^{(j)} \right\| \; 0(h^{2r-j}), \quad 0 \le j \le 2r-1, \tag{5.4}$$

and, because $y_1^{(2j+1)}(x_i) = s_i^{(2j+1)}, 0 \le j \le r-2,$, the odd derivatives of s display superconvergence at the knots in the sense that

$$y_i^{(2j+1)} = s_i^{(2j+1)} + 0(h^{2r-2j}), \quad 1 \le j \le 2r-2; \tag{5.5}$$

see Remark 2.5. Thus,

$$y_i^{(j)} = s_i^{(j)} + 0(h^{kj}), \quad 1 \le j \le 2r-3, \tag{5.6}$$

where $k_j \geq 4$ for both r = 2 and r = 3. This means that in this case $k_j \geq 4$ in (3.4). Also, because $P_i^{(2r-2)}$ (0) = 0, Theorem 2,1 gives

$$y_i^{(2r-2)} = Y_{1*}^{(2r-2)}(x_i) + 0(h^4); \qquad i = 0,1,\ldots,n-1, \tag{5.7}$$

where Y_{1*} denotes the corrected spline approximation Y_1 with $d_{0,1}^{(2r)}$ replaced by $d_{0,2}^{(2r)}$; see Remark 2.6 (i).

In the extrapolated collocation method of Daniel and Swartz [6] the approximating spline 's' is obtained by replacing the derivatives $y_i^{(j)}$, $1 \le j \le$ 2r-2, in (3.2) by the approximations contained in (5.6) and (5.7). This follows from the discussion of Section 3, by observing that the approximations (5.7) give precisely the replacements (3.6). In other words the method of [6] may be regarded as a collocation scheme, where the collocation is performed by means of the corrected spline approximation Y_{1*}

It follows that in (3.12a) $\ell = 4$ and that the matrix (3.12c) is

$$A_n = I + \Lambda_{n,2r-2}, \tag{5.8a}$$

where $\Lambda_{n,2r-2}$ has the partitioned form

$$\Lambda_{n,2r-2} = \frac{1}{12}\left[\begin{array}{c|c|c} 2 & \underline{a}^T & 0 \\ \hline \underline{b} & T_n & \underline{b}_R \\ \hline 0 & \underline{a}_R^T & 2 \end{array}\right]$$

(5.8b)

with $\underline{a}$, $\underline{b}$, $\underline{a}_R$, $\underline{b}_R$ and T_n as described below:

(i) $\underline{a}$ and $\underline{b}$ are respectively the (n - 1)-dimensional column vectors $\underline{a} = (-5,4,-1,0,...,0)^T$ and $\underline{b} = (1,0,\ldots 0)^T$.

(ii) $\underline{a}_R$ and $\underline{a}_R$ are the vectors $\underline{a}$ and $\underline{b}$ with their components written down in the reverse order, (Throughout the paper, if $\underline{v} = (v_1, v_2, ..., v_{n-1})^T$ we use $\underline{v}_R$ to denote the vector $\underline{v}_R = (v_1, v_2, ..., v_{n-1})^T$.)

(iii) $T_n = [t_{ij}]$ is the $(n-1)\times(n-1)$ tri-diagonal matrix with $t_{ii} = -2$ and $t_{i,i-1} = t_{i,i+1} = 1$.

Clearly, the matrix A_n given by (5.8) is uniformly bounded. Also, it can be shown easily that $\left\|A_n^{-1}\right\| \le 1.86$, and that for each fixed $u \in c[a,b]$

$\lim\limits_{n\to\infty} \left\|D_n u - A_n D_n u\right\| = 0$ where D_n is the restriction operator (4.5); see [6: pp 166-167]

Therefore, the conditions A4.4 and A4.5 hold, and (4.13) gives

$$\left\|\tilde{s}^{(j)} - y^{(j)}\right\| = 0\left(h^{\gamma}\right), \qquad 0 \le j \le 2r-2 \tag{5.9a}$$

where

$$\gamma = \min\{2r - j, 4\}.$$

As we indicated in Section 4, the above method of analysis leading to the result (5.9) is due to Daniel and Swartz [6], who proposed and analyzed the extrapolated collocation method for the case r = 2 only; i.e. the cubic spline solution of second-order problems. For the case r=2, Daniel and Swartz also proved that the derivatives of the cubic spline $\tilde{s}$ display superconvergence at the points (2.17), and described a method for computing improved approximations to $y^{(j)}, 1 \le j \le 4$, at any point in [a,b]. This method of [6] consists of constructing a piecewise quartic, q , given in terms of the values $\tilde{s}_i^{(1)}, \tilde{s}_{i+\frac{1}{2}}^{(1)}; i = 0,1,\ldots, n-1$,

and having breakpoints in $\left\{x_i, x_{i+\frac{1}{2}}\right\}_{i=0}^{n}$, so that $\| q^{(J)} - y^{(J)} \| = 0(h^{5-J})$, $1 \le j \le 4$;

see [6: Cor. 4.15]. Alternatively, improved approximations to $y^{(j)}$ can be obtained as indicated below, by constructing corrected approximations of the form (2.1) - (2.7).

Let $\tilde{y}_M$ denote the corrected approximations obtained from (2.1) by replacing s by $\tilde{s}$ and $d_{i,M}^{(2r+m)}$ by $\tilde{d}_{i,M}^{(2r+m)}$, where $\tilde{d}_{i,M}^{(2r+m)}$ denote the derivative approximations (2.4)-(2.7)corresponding to the spline $\tilde{s}$. Then , it follows from Theorm 2.1 that, for $1 \le M \le 2$,

$$\begin{aligned} y^{(j)} - \tilde{y}_M^{(j)} &= y^{(j)} - y_M^{(j)} + y_M^{(j)} - \tilde{y}_M^{(j)} \\ &= y_M^{(j)} - \tilde{y}_M^{(j)} + 0\left(h^{2r-j+M}\right), 0 \le j \le 2r, \end{aligned} \tag{5.10a}$$

Where

$$y_M^{(j)} - \tilde{y}_M^{(j)} = s^{(j)} - \tilde{s}^{(j)} + \sum_{m=0}^{M-1} \left\{ \frac{h^{2r-j+m}}{(2r+m)!} \left(d_{i,M}^{(2r+m)} - \tilde{d}_{i,M}^{(2r+m)} \right) p_m^{(j)} \right\} \tag{5.10b}$$

and where we used the abbreviations $y_M^{(j)}$ for $y_M^{(j)}$, e. t. c. Also, since $\beta = \min\{k, \ell\} = 4$, (4.12) gives

$$\left\| s^{(j)} - \tilde{s}^{(j)} \right\| = 0\left(h^4\right), 0 \le j \le 2r-2, \tag{5.11a}$$

and hence

$$\left\| s^{(2r-1)} - \tilde{s}^{(2r-1)} \right\| = 0\left(h^3\right). \tag{5.11b}$$

Furthermore, (5.11a) and the definitions of $d_{i,M}^{(2r+m)}$ and $\tilde{d}_{i,M}^{(2r+m)}$ imply that

$$d_{i,M}^{(2r+m)} = d_{i,M}^{(2r+m)} \text{and } d_{i,M}^{(2r+m)} = d_{i,M}^{(2r+m)} + 0(h) \; ; i=0,1,\ldots,n-1. \tag{5.12}$$

Finally, by combining the results (5.10) - (5.12) we find that for $1 \le M \le 2$, $0 \le \mu \le 1$ and $0 \le j \le 2r$,

$$y^{(j)}(x_i + \mu h) = \tilde{y}_M^{(j)}(x_i + \mu h) + 0\left(h^{\nu}\right); i = 0,1,\ldots, n-1, \tag{5.13a}$$

Where

$$\nu = \min\{2r - j + M, 4\} \tag{5.13b}$$

This implies the following:

(i) In the case r = 2, the best order of approximation to y is that given by (5.9), i.e.

$$y(x) = \tilde{s}(x) + 0(h^4), \quad x \in [a,b]. \tag{5.14}$$

For the derivatives of y however improved orders of approximation can be obtained by using one or two correction terms as follows:

(a) For $0 \leq \mu \leq 1$,

$$y^{(1)}(x_i + \mu h) = \tilde{y}_1^{(1)}(x_i + \mu h) + 0(h^4); \quad i = 0,1,\ldots\ldots,n-1 \tag{5.15}$$

(b) For $0 \leq \mu \leq 1$ and $2 \leq j \leq 4$,

$$y^{(j)}(x_i + \mu h) = \tilde{y}_2^{(j)}(x_i + \mu h) + 0(h^{6-j}); \quad i = 0,1,\ldots\ldots,n-1) \tag{5.16}$$

(ii) In the case r=3, the best orders of approximation to y, $y^{(1)}$ and $y^{(2)}$ are those given by (5.9), i.e. for $0 \leq j \leq 2$

$$y^{(j)}(x) = \tilde{s}^{(j)}(x) + 0(h^4), \qquad x \in [a,b] \tag{5.17}$$

But, for higher derivatives improved orders of approximation can be obtained as follows:

(a) For $0 \leq \mu \leq 1$,

$$y^{(3)}(x_i + \mu h) = \tilde{y}_2^{(3)}(x_i + \mu h) + 0(h^4); \quad i = 0,1,\ldots\ldots,n-1 \tag{5.18}$$

(b) For $0 \leq \mu \leq 1$ and $4 \leq j \leq 6$,

$$y^{(j)}(x_i + \mu h) = \tilde{y}_2^{(j)}(x_i + \mu h) + 0(h^{8-j}); \quad i = 0,1,\ldots\ldots,n-1 \tag{5.19}$$

5.3 A $0(h^6)$ quintic spline method for fourth-order problems.

Here we consider only the case r = 3, and assume that the boundary conditions (3.1b) involve only function values and first derivatives. That is we consider the use of quintic splines for the solution of fourth-order linear boundary-value problems with homogeneous boundary conditions of the form

$$\mathcal{B}y := y(a) = y(b) = y^{(1)}(a) = y^{(1)}(b) = 0. \tag{5.20}$$

We also assume that $y \in c^{10}[a,b]$ and that the end conditions of the interpolating quintic spline s are of order $p \geq 6$. Then

$$\left\| y^{(j)} - s^{(j)} \right\| = 0(h^{6-j}), 0 \leq j \leq 5, \tag{5.21}$$

and

$$y_i^{(1)} = s_i^{(1)} + 0(h^6); \quad i = 0,1,\ldots\ldots, n; \tag{5.22}$$

see Remark 2.5. This implies that $\mathcal{B}s = 0(h^6)$, i.e. k=6 in (3.4). In addition, our continuity and end-condition assumptions imply the following:

(i) $$y_i^{(2)} = y_1^{(2)}(x_i) + 0(h^6); \quad i = 0,1,\ldots\ldots, n-1 \tag{5.23}$$

where Y_{1*} denotes the corrected quintic spline approximation in equation (2.21) of Remark 2.6(i).

(ii) $$y_i^{(3)} = y_1^{(3)}(x_i) + 0(h^6); \quad i = 0,1,\ldots\ldots, n-1 \tag{5.24}$$

where Y_{2*} denotes the corrected quintic spline approximation in equation (2.22) of Remark 2.6 (ii).

(iii) $$y_i^{(4)} = y_1^{(4)}(x_i) + 0(h^6); \quad i = 0,1,\ldots\ldots, n-1 \tag{5.25}$$

where Y_{3*} denotes the corrected quintic spline approximation in equation (2.25) of Remark 2.7 .

In the method under consideration, the approximations contained in (5.22) and (5.23)-(5.25) are used to replace respectively the derivatives $y_i^{(1)}$ and $y_i^{(j)}$, $2 \le j \le 4$ (3.2). (The approximations needed for replacing the derivatives $y_n^{(j)}$, $2 \le j \le 4$ can of course be deduced immediately from the corresponding approximations to $y_0^{(j)}$ Then, the exponent in (3.12a) is $\ell = 6$, and the matrix $\mathcal{A}_n$ has the form

$$\mathcal{A}_{n=I} + \square_{n,4} + \Delta_{n,3}\, \Delta_{n,3} + \Delta_{n,2}\, \square_{n,2}, \tag{5.26a}$$

where:

(i) $\Delta_{n,2}$ and $\Delta_{n,3}$ are the diagonal matrices defined by (3.12b).

(ii) The matrix $\Delta_{n,4}$ has the partitioned form

$$\Delta_{n,4} = \frac{1}{240}\left[\begin{array}{c:c:c:c:c} 77 & -126 & \underline{c}^T & 0 & 0 \\ \hdashline 18 & -31 & \underline{d}^T & 0 & 0 \\ \hdashline \underline{e} & \underline{g} & \underline{Q}_n & \underline{g}_R & \underline{e}_R \\ \hdashline \underline{0} & 0 & \underline{d}_R^T & -31 & 18 \\ \hdashline 0 & 0 & \underline{c}_R^T & -266 & 77 \end{array}\right] \tag{5.26b}$$

Where $\underline{c}, \underline{d}, \underline{e}$ and $\underline{g}$ are the (n - 3)-dimensional column vectors, $\underline{c} = (374,-276,109,-18,0,\ldots,0)^T$, $\underline{d} = (4,14,-6,1,0,\ldots,0)^T$, $\underline{e} = (-1,0,\ldots.,0)^T$, $\underline{g} = (24,-1,0,\ldots,0)^T$, and $Q_n = [q_{ij}]$ is the (n-3) - (n - 3) quindiagonal matrix with $q_{ij} = -46$, $q_{ij} = 24$ for $|i-j| = 1$, and $q_{ij} =$ -for $|i-j|=2$.

(iii) The matrix $\square_{n,3}$ has the partitioned form

$$\Delta_{n,3} = \frac{h}{480}\left[\begin{array}{c|c|c|c|c} -5 & 18 & \underline{u}^T & 0 & 0 \\ \hline -3 & 10 & \underline{v}^T & 0 & 0 \\ \hline \underline{x} & \underline{y} & \underline{U}_n & -\underline{y}_R & -\underline{x}_R \\ \hline 0 & 0 & -\underline{v}_R^T & -10 & 3 \\ \hline 0 & 0 & -\underline{u}_R^T & -18 & 5 \end{array}\right] \tag{5.26c}$$

Where $\underline{u}, \underline{v}, \underline{x}$ and $\underline{y}$ are the (n - 3)-dimensional column vectors $\underline{u} = (-24,14,-3,0,..,,0)^T$, $\underline{v} = (-12,6,-1,0,\ldots,0)^T$, $\underline{x} = (-1,0,\ldots,0)^T$, $\underline{y} = (2,-1,0,\ldots,0)^T$, and $U_n = [u_{ij}]$ is the (n-3) x (n-3) quindiagonal matrix with $u_{ii} = 0, u_{i,i-1} = 2, u_{i,i+1} = -2$, $u_{i,i-2} = -1$ and $u_{i,i+2} = 1$

(iv) The matrix $\square_{n,2}$ is a scalar multiple of the matrix (5.8b), i.e.

$$\Delta_{n,2} = -\frac{h^2}{60}\,\Delta_{n,2r-2}, \tag{5.26d}$$

where $\Delta_{n,2r-2}$ is the matrix defined by (5.8b).

The matrix A_n given by (5.26) is clearly uniformly bounded. Therefore, in order to establish the convergence of the method we have to show that, for sufficiently large n, A_n has a uniformly bounded inverse A_n^{-1}, and that for each fixed u □ c[a,b]

$$\lim_{n \to \infty} \left\| D_n u - A_n D_n u \right\| = 0, \tag{5.27}$$

where D_n is the restriction operator (4.5). The first of these can be proved as follows.

Let

$$B_n = I + \Delta_{n,4} \tag{5.28}$$

Then, by performing a number of elementary row operations it is easy to show that B_n is invertible and that $\|B_n^{-1}\| \le 6.25$. This means that the matrix A_n can be written as

$$A_n = B_n \{I + B_n^{-1}(\Delta_{n,3}\Delta_{n,3} + \Delta_{n,2}\Delta_{n,2})\} = B_n \{I + C_n\}, \tag{5.29a}$$

Where

$$\begin{aligned} \|c_n\| &\le \|B_n^{-1}\| \left(\|\Delta_{n,3}\| \; \|\Delta_{n,3}\| + \|\Delta_{n,2}\| \; \|\Delta_{n,2}\| \right) \\ &\le 6.25 \; (8h \, \| e_3 \| / 60 + h^2 \, \| e_2 \| / 60). \end{aligned} \tag{5.29b}$$

Thus, for sufficiently large n , $\|C_n\| < 1$ and hence, from (5.29a), the matrix A_n possesses a uniformly bounded inverse A_n^{-1}.

To prove (5.27), let

$$\begin{aligned} \underline{z}^{\{n\}} &:= D_n u - A_n D_n u \\ &= \left(z_i^{\{n\}} \right)_{i=0}^{n} \end{aligned} \tag{5.30}$$

Then,

$$\begin{aligned} \left| z_0^{\{n\}} \right| \le & \left| -77u_0 + 266u_1 - 376u_3 - 106_4 + 18u_5 \right| / 240 \\ & + h\left| e_3(a) \right| \left| 5u_0 - 18u_1 + 24u_3 + 3u_4 \right| / 480 \\ & + h^2 \left| e_2(a) \right| \left| 2u_0 - 5u_1 + 4u_2 + u_3 \right| / 720 \end{aligned}$$

i.e.

$$\begin{aligned} \left| z_0^{\{n\}} \right| \le \{ & 280\omega(u;5h) + 8h\left| e_3(a) \right| \omega(ul4h) \\ & + h^2 \left| e_2(a) \right| \omega(u;3h) \} / 60 \end{aligned} \tag{5.31}$$

where ω(u ; h) denotes the modulus of continuity of u over an interval of width h. Therefore,

$$\lim_{n \to \infty} \left| z_i^{\{n\}} \right| = 0.$$

In exactly the same manner it can be shown that

$$\lim_{n \to \infty} \left| z_i^{\{n\}} \right| = 0; i = 1,2,.....,n.$$

Thus, the conditions A4.4 and A4.5 hold, and (4.13) gives

$$\left\|\tilde{s}^{(J)} - y^{(j)}\right\| = 0(h^{6-j}), 0 \le j \le 4. \quad (5.32)$$

Also, by modifying in an obvious manner the analysis leading to the result (5.13) , it is easy to show that for $1 \le M \le 3$, $0 \le \mu \le 1$ and $0 \le j \le 6$,

$$y^{(j)}(x_i + \mu h) = \tilde{Y}_M^{(j)}(x_i + \mu h) + 0(h^{\nu}); \quad i = 0,1,\dots,n-1, \quad (5.33a)$$

where

$$\nu = \min\{6 - j+,6\}, \quad (5.33b)$$

and where $\tilde{Y}_M$ denote the corrected approximations corresponding to the quintic spline $\tilde{S}$.

Remark 5.1 The requirement that the boundary conditions are of the form (5.20) is needed for the application of the convergence analysis of Section 4. However, it is reasonable to expect that the same convergence results will hold when the boundary conditions are of the general form (3.1b), provided that the second and third derivatives in (3.1b) are replaced by the appropriate corrected approximations given by (5.23) and (5.24). This conjecture is supported by the results of Example 8.3 considered in Section 8.

6. A $0(h^6)$ quintic spline method for linear second-order boundary-value problems.

In this section we describe a $0(h^6)$ modified collocation method for the solution of second-order boundary-value problems of the form

$$L[y] : y^{(2)}(X) + e_1(x)y^{(1)}(x) + e_0(x)y(x) = f(x), \quad x \in [a,b], \quad (6.1a)$$

$$By = 0, \quad (6.1b)$$

where (6.1b) denotes two boundary conditions of the form

$$\alpha_{i0} y(a) + \beta_{i0} y(b) + \alpha_{i1} y^{(1)}(b) = 0; \quad i = 0,1 \quad (6.1b)$$

That is the problems under consideration are of the form (3.1) with r = 2. Here however the approximating spline $\tilde{s}$ s is taken to be quintic rather than cubic. For this reason, the resulting method is not of the type described in Section 3.

Assume that the conditions A 4.1 -A4.3, concerning the functions e_0, e_1, f and the solution y of (6.1) hold with m = 8 in A4.2, and let s be a $S_5(\pi_n)$,-interpolate of y. Assume also that the end conditions of s are of order $p \geq 4$, and let $\{t_i\}$ denote the following n +3 points:

$$t_0 = x_0 + h/2, \quad t_i = x_{i-1}; \quad i = 1,2,....,n+1, \quad t_{n+2} = x_n - h/2. \qquad (6.2)$$

Then, our assumptions that $y \in C^8\ [a,b]$ and $p \geq 4$ imply that:

$$\left\| s^{(j)} - y^{(j)} \right\| = 0(h^{6-j}), \quad 0 \leq j \leq 5, \qquad (6.3)$$

$$y^{(1)}(t_i) = s^{(1)}(t_i) + 0(h^6); \qquad i = 0,1,...,n+2, \qquad (6.4)$$

and

$$y^{(2)}(t_i) = Y_2^{(2)}(t_i) + 0(h^6); \qquad i = 0,1,...,n+2; \qquad (6.5)$$

see Theorem 2.1 and Remark 2.5.

The method of this section is based on substituting the derivatives in the boundary conditions (6.1b) and in the n + 3 equations

$$L[y](t_i) = f(t_i); \quad i = 0,1,...,n+2, \qquad (6.6)$$

by the approximations given by (6.4) - (6.5). Here however, we express the two corrections in Y_2 in terms of the second derivatives $s_i^{(2)}$ of s. (That is we take the values $d_{i,2}^{(6)}$ and $d_{i,2}^{(7)}$ involved in the corrected approximation Y_2 to be those given by (2.29) - (2.30) and (2.6); see Remark 2.8.) The above replacements then lead to the equations:

$$Bs = 0(h^6), \qquad (6.7)$$

and

$$A_n \underline{s}^{(2)} + \Delta_{n,1} \underline{s}^{(1)} + \Delta_{n,} \underline{s} = \underline{f} + \underline{0}(h^6), \qquad (6.8a)$$

where $\underline{s}^{(j)}$, $\underline{f}$, $\Delta_{n,j}$ and A_n are as follows;

(i) $\underline{s}^{(2)}$, $\underline{s}^{(1)}$ and $\underline{f}$ are the (n +3)-dimensional column vectors,

$$\underline{s}^{(j)} = \left\{ s^{(j)}(t_i) \right\}_{i=0}^{n+2}, \quad 1 \leq j \leq 2, \text{ and } \underline{f} = \left\{ f(t_i) \right\}_{i=0}^{n+2} \qquad (6.8b)$$

(ii) $\Delta_{n,j}$, $0 \leq j \leq 1$, are the (n + 3) × (n + 3) diagonal matrices

$$\Delta_{n,j} = \operatorname{diag}\left\{ e_j(t_0), e_j(t_1), ...e_j(t_{n+2}) \right\}. \qquad (6.8c)$$

(iii) The matrix A_n has the partitioned, form

$$A_n = \frac{1}{720}\left[\begin{array}{c|c|c|c|c|c|c}
720 & 35/16 & -161/16 & \underline{a}^T & 0 & 0 & 0 \\ \hline
0 & 717 & 14 & \underline{b}^T & 0 & 0 & 0 \\ \hline
0 & -2 & 729 & \underline{c}^T & 0 & 0 & 0 \\ \hline
\underline{o} & \underline{e} & \underline{g} & Q_n & \underline{g}_R & \underline{e}_R & \underline{o} \\ \hline
0 & 0 & 0 & \underline{c}_R^T & 729 & -2 & 0 \\ \hline
0 & 0 & 0 & \underline{b}_R^T & 14 & 717 & 0 \\ \hline
0 & 0 & 0 & \underline{a}_R^T & -161/16 & 35/16 & 720
\end{array}\right] \qquad (6.8d)$$

where $\underline{a}, \underline{b}, \underline{c}, \underline{e}$ and $\underline{g}$ are the (n - 3)-dimensional column vectors

$\underline{a} = \frac{1}{16}(294,-266,119,21,\ldots,0)^T$, $\underline{b} = (-26, 24, -11, 2, 0, \ldots 0)^T$, $\underline{c} = (-16, 14, -6, 1, 0, \ldots. 0)^T$, $\underline{e} = (-1, 0, \ldots., 0)^T$ and $\underline{g} = (4, -1, 0, \ldots., 0)^T$, $\underline{o}$ is the $(n-3)-$ dimensional null vector, and $Q_n = [q_{ij}]$ is the (n-3) x (n-3) quindiagonal matrix with $q_{ii} = 714$, $q_{ij} = 4$ for $|i-j| = 1$, and $q_{ij} = -1$ for $|i-j| = 2$.

By analogy with the work of the previous sections, we consider now the problem of determining an approximating quintic spline $\tilde{s}$ by simply dropping the $0(h^6)$ terms from the equations (6,7) and (6.8), That is, we consider a modified collocation method for the solution of problems of the form (6.1), where the approximating quintic spline $\tilde{s}$ is defined by the equations

$$A_n \underline{\tilde{s}}^{(2)} + \Delta_{n,1}\underline{\tilde{s}}^{(1)} + \Delta_{n,0}\underline{\tilde{s}} = \underline{f}\ , \qquad (6.9a)$$

$$B\tilde{s} = 0. \qquad (6.9b)$$

It turns out that the convergence of this method can also be established by the analysis of Daniel and Swartz [6: §4], provided that their restriction and prolongation operators (4.5) and (4.6) are re-defined as follows:

(i) $$D_n :\ C[a,b] \to IR_{n+3}, \qquad (6.10a)$$

where for any $g \in C[a,b]$,

$$(D_n g)_i = g(t_i); \quad i = 0,1,\ldots,n+2. \tag{6.10b}$$

(ii) $$M_n : IR_{n+3} \to s_3\ (\pi_n), \tag{6.11a}$$

where for any vector $\underline{z} \in IR_{n+3}$, $M_n\underline{z}$ is the cubic spline $w \in S_3\ (\pi_n)$ satisfying the (n+3) interpolation conditions

$$w(t_i) = z_i; \qquad i = 0,1,\ldots,n+2. \tag{6.11b}$$

(It is well-known that w: = $M_n\underline{z}$ exists uniquely for any vector $\underline{z} \in IR_{n+3}$; see e.g. [5], [8: p.577] and [12: p.25].)

With these new definitions of D and M we can proceed exactly as in [6: §4], and thus conclude that the modified collocation method (6.9) is well-defined, and that the second derivative $\tilde{s}^{(2)}$ of the resulting quintic spline $\tilde{s}$ converges uniformly to $y^{(2)}$. More precisely, the above results can be deduced immediately from the analysis of [6: pp.167-68], by observing the following:

(a) The boundary value problem (6.1) and the approximating equations (6.9) can be written in equivalent operator forms as

$$(I+K)v = f, \qquad v:= y^{(2)}, \tag{6.12}$$

and

$$A_n D_n v_n + D_n K v_n = D_n f, \qquad v_n := \tilde{s}^{(2)}, \tag{6.13}$$

where A_n is the matrix (6.8d), D_n is the restriction operator (6.10), and K is the operator (4.7) with r = 2.

(b) The prolongation operator M_n defined by (6.11) is uniformly bounded. This can be proved easily by using standard cubic spline results.

(c) $D_n M_n = I_n$, where I_n is the identity operator $I_n : IR_{n+3} \to IR_{n+3}$.

(d) The matrix A_n is uniformly bounded. It is also strictly diagonally dominant, and hence invertible with $\|A_n^{-1}\| \le 1.09$.

(e) For each fixed $u \in C\,[a,b]$, $\lim_{n\to\infty} \|D_n u - A_n D_n u\| = 0$. This can be established as indicated in Section 5.3, by letting $\underline{z}^{\{n\}} := D_n u - A_n D_n u$ and showing that $|z_i^{\{n\}}| \le$ const, $\times\ \omega$ (u; 5h) ; i = 0,1,... ,n + 2.

(f) Because of (d) and (e), the matrix A_n can take the place of the matrix Q_n involved in the analysis of [6].

The above observations also imply that

$$\left\|\tilde{s}^{(j)} - s^{(j)}\right\| = 0(h^{6}), \quad 0 \le j \le 2, \tag{6.14a}$$

and hence that

$$\left\|\tilde{s}^{(j)} - s^{(j)}\right\| = 0(h^{8-j}), \quad 3 \le j \le 5. \tag{6.14b}$$

This can be proved by modifying in an obvious manner the analysis of [6: p.168], Therefore, from (6.3) and (6.14) we have that

$$\left\|\tilde{s}^{(j)} - y^{(j)}\right\| \; 0(h^{6-j}), \quad 0 \le j \le 5. \tag{6.15}$$

Finally, improved approximations to the derivatives of y can be obtained from the corrected approximations $\tilde{Y}_M, 1 \le M \le 2$, corresponding to the quintic spline $\tilde{s}$ in exactly the same manner as in Section 5. The precise result in this case is that, for $1 \le M \le 2$, $0 \le \mu \le 1$ and $0 \le j \le 6$,

$$y^{(j)}(x_i + uh) = \tilde{Y}_M^{(j)}(x_i + \mu h) + 0(h^{\nu}); \qquad i = 0,1,....,n-1, \tag{6.16a}$$

Where

$$\nu = \min\{6 - j + M, 6\}. \tag{6.16b}$$

Remark 6.1 Let $y \in C^6[a,b]$, assume that the end conditions of the interpolating quintic spline s are of order $p \ge 2$, and let the corresponding approximating spline $\tilde{s}$ be determined by standard collocation at the n + 3 points (6.2). Then, the equations (6.7), (6.8), (6.9a) and (6.9b) simplify respectively as follows:

$$\mathcal{B}s = 0(h^{k}), \qquad k \ge 5, \tag{6.17}$$

$$\underline{s}^{(2)} + \Delta_{n,1}\underline{s}^{(1)} + \Delta_{n,0}\underline{s} = \underline{f} + 0(h^{4}) \tag{6.18}$$

$$\underline{\tilde{s}}^{(2)} + \Delta_{n,1}\underline{\tilde{s}}^{(1)} + \Delta_{n,0}\underline{\tilde{s}} = \underline{f}, \tag{6.19a}$$

and

$$B\tilde{s} = 0. \tag{6.19b}$$

That is, the $0(h^6)$ terms in (6.7) and (6.8) are replaced respectively by $0(h^k), k \ge 5,$ and $0(h^4)$, and the matrix A_n in (6.7) and (6.9a) is replaced by the identity matrix. Because of this, the convergence of the collocation method defined by (6.19) can be deduced immediately from the analysis outlined above. The precise result in this case is that

$$\left\|\tilde{s}^{(j)} - y^{(j)}\right\| = 0(h^{\gamma}), \qquad 0 \le j \le 5, \tag{6.20a}$$

where

$$\gamma = \min\{6 - j, 4\}; \tag{6.20b}$$

see [10] and [12: p.21].

7. Nonlinear problems.

In this section we indicate how some of the results of the previous sections can be extended to nonlinear problems of the form

$$y^{(2r-2)}(x) = f(x, y(x), \ldots y^{(2r-3)}(x)), \qquad x \in [a,b], \tag{7.1a}$$

$$By := \sum_{j=0}^{2r-3} \{\alpha_{ij} y^{(j)}(a) + \beta_{ij} y^{(j)}(b)\} = 0, \quad 0 \le i \le 2r-3, \tag{7.1b}$$

where as before r = 2 or r = 3.

As is well-known a solution of (7.1) is not necessarily unique. For this reason, we only consider modified collocation methods when applied to a sufficiently small neighbourhood of an isolated solution. Furthermore, for the purposes of the analysis we only consider modified collocation methods in which the derivatives $y^{(j)}$, $1 \le j\, 2r-3$, in the nonlinear part of (7.1a) are replaced by the corresponding spline derivatives. That is, we assume that the approximating spline $\tilde{s} \in s_{2r-1}\ (\pi_n)$ is defined by a nonlinear system of the form

$$A_n \underline{\tilde{s}}^{(2r-2)} = \underline{f}, \qquad B\tilde{s} = 0, \tag{7.2a}$$

where $\underline{f}$ is the (n +1)-dimensional column vector

$$\underline{f} = \left\{ f\left(x_i, \tilde{s}_i, \ldots, \tilde{s}_i^{(2r-3)}\right) \right\}_{i=0}^{n}, \tag{7.2b}$$

and

$$A_n = I + \Lambda_{n,2r-2}, \tag{7.2c}$$

where the matrix $\Lambda_{n,2r-2}$ has the same meaning in Section 3. (Of course, we also assume that (7.2) is derived by dropping the $\underline{0}(h^{\ell})$ and $0(h^k)$ terms from the equations

$$A_n \underline{s}^{(2r-2)} = \underline{f} + \underline{0}(h^{\ell}), \qquad Bs = 0(h^k), \tag{7.3}$$

corresponding to the interpolatory spline $s \in S_{2r-1}\ (\pi_n)$.) Then, the analysis reduces essentially to that of Daniel and Swartz [6 : §5], which in turn is based closely on the convergence analysis of Russell and Shampine [13 : §4], The main details are as follows.

In place of the assumptions A4.1-A4.3, we now assume the following in connection with the boundary value problem (7.1):

A7,1. The boundary value problem (7.1) has at least a solution $y \in C^m[a,b]$, $m \geq 2r$, and the function f is sufficiently smooth near y in the sense that $f \in C^2[\overline{N}]$, where $\overline{N}$ is a neighbourhood of the "curve" $\{[x, y(x), \ldots, y^{(2r-3)}(x)]^T :$ $x \in [a,b]\ \}$ **see** [4 : p.590].

A7.2 The equation $u^{(2r-2)}(x) = 0$ with boundary conditions $\mathcal{B}u$ - 0 has only the trivial solution.

A7.3 The equation

$$u^{(2r-2)}(x) - \sum_{j=0}^{2r-3} \frac{\partial f(x, y, \ldots y^{(2r-3)})}{\partial z_j}\ u^{(j)}(x) = 0, \tag{7.4}$$

subject to the boundary conditions $\mathcal{B}u = 0$, has only the trivial solution. (Here y stands for the solution referred to in A7.1.)

Regarding the matrix $\mathcal{A}_n$ in (7.2), we assume that this matrix satisfies precisely the same conditions A4.4 and A4.5 as in the linear case. Then, the following results can be proved by modifying in an obvious manner the analysis of [6: §5].

(i) There exists a $\sigma > 0$ such that there is no other solution $\hat{y}$ of (7.1) satisfying

$$\left\| y^{(2r-2)} - \hat{y}^{(2r-2)} \right\| \leq \sigma. \tag{7.5}$$

(Here also y stands for the solution referred to in A7.1).

(ii) For sufficiently large n there exists a unique spline $\tilde{s} \in S_{2r-1}(\pi_n)$ solving the equations (7.2) and satisfying

$$\left\| y^{(2r-2)} - \tilde{s}^{(2r-2)} \right\| \leq \sigma \tag{7.6}$$

(iii) The spline $\tilde{s}$ satisfies

$$\left\| \tilde{s}^{(j)} - y^{(j)} \right\| = 0(h^{\gamma}); \qquad 0 \leq j \leq 2r-2, \tag{7.7a}$$

where as in (4.13),

$$\gamma = \min \{2r - j, k, \ell\}, \tag{7.7b}$$

and ℓ,k, are the orders of approximation in (7.3).

We end this section by making the following remarks concerning the application of specific modified collocation methods to nonlinear boundary value problems.

Remark 7.1. *Standard collocation*

In this method all the derivatives $y_i^{(j)}$ are replaced by the corresponding spline derivatives $s_i^{(j)}$ Therefore, the analysis outlined above applies and, as in the linear case, the result (5.3) holds; see [13 : §4].

□

Remark 7.2 *The extrapolated collocation method of Daniel and Swavts [6].*

Since only the derivatives $y_i^{(2r-1)}$ are replaced by linear combinations of the $\tilde{s}_i^{(2r-2)}$, the analysis also applies directly to this method. Therefore, the result (5.9) holds for nonlinear problems of the general form (7.1). Furthermore, the result (5.13) concerning the quality of the corrected approximations $\tilde{Y}_M^{(j)}$ can be established exactly as in the linear case.

□

Remark 7.3 *The 0(h^6) quintic spline method of Section 5.3.*

In this case the analysis applies directly to fourth-order nonlinear problems of the form

$$y^{(4)}(x) = f(x, y(x), y^{(1)}(x)), \qquad x \in ia, b], \tag{7.8a}$$

$$y(a) = y(b) = y^{(1)}(a) = y^{(1)}(b) = 0 \tag{7.8b}$$

This shows that the results (5.32) and (5.33) also hold for nonlinear problems of the form (7.8). In fact, it is easy to see that, the same convergence results hold for problems of the more general form,

$$y^{(4)}(x) + e_3(x)y^{(3)}(x) + e_2(x)y^{(2)}(x) = f(x, y)(x), y^{(1)}(x)), \qquad x \in [a, b], \tag{7.9a}$$

$$y(a) = y(b) = y^{(1)}(a) = y^{(1)}(b) = 0. \tag{7.9b}$$

□

Remark 7.4. *The 0(h^6) quintic spline method of Section 6.*

The analysis does not apply directly in this case. However, by modifying the arguments in the manner indicated in Section 6, it is easy to show that the results

(6.15) and (6.16) hold for second-order boundary value problems of the general form

$$y^{(2)}(x) = f(x, y), y^{(1)}(x)), \qquad x \in [a, b]. \tag{7.10a}$$

$$By = 0. \square \tag{7.10b}$$

Remark 7.5 Although the operators P_n defined by (4.11) are not projectors, the convergence results of this section can also be established by modifying the analysis contained in Section 3 of the paper by de Boor and Swartz [4]; see the remarks in p.606 of [4] and p.170 of [6]. This alternative analysis can also be used to show that the Newton iterative method applied to the equations (7,2) converges locally to $\tilde{s}^{(2r-2)}$ at a quadratic rate. The application of the iterative method for computing successive approximations $\tilde{s}_{(k)}$; $k = 0,1,\ldots$ to $\tilde{s}$ may be described as follows:

"Given $\tilde{s}_{(k)}$, find the modified collocation approximation $\tilde{s}_{(k+1)}$ to the solution ω of the linear boundary value problem

$$\omega^{(2r-2)} - \sum_{j=0}^{2r-3} \frac{\partial f\left(x, \tilde{s}_{(k)}, \ldots, \tilde{s}_{(k)}^{(2r-3)}\right)}{\partial z_j} \omega^{(j)}$$

$$= f\left(x, \tilde{s}_{(k)}, \ldots, \tilde{s}_{(k)}^{(2r-3)}\right) - \sum_{j=0}^{2r-3} \frac{x, \tilde{s}_{(k)}, \ldots, \tilde{s}_{(k)}^{(2r-3)}}{\partial z_j} \tilde{s}_{(k)}^{(j)}, \tag{7.11a}$$

$$B\omega = 0; \tag{7.11b}$$

See [4; p.594]."

8. Numerical examples

In this section we present the results of several numerical examples, illustrating the theory of previous sections. These results were computed on an Eclipse MV/8000 computer, using programs written in double-precision Fortran; i.e. a precision of between 16 and 17 significant figures. Our programs were based on representing the approximating spline $\tilde{s}$ in terms of B-splines, and made extensive use of the B-spline subroutines of de Boor [2]; see also [3].

As before, let $\widetilde{Y}_M$, $M>0$, denote the corrected approximations corresponding to the spline $\widetilde{s}$ Also, let $\widetilde{Y}_0 := \widetilde{s}$, i.e. let $\widetilde{Y}_0$ denote the modified collocation solution of the boundary-value problem under consideration. Then, the results listed in the tables are estimates of the uniform norms $\left\| y^{(j)} - \widetilde{Y}_M^{(j)} \right\|$, $M \le 0$, obtained by sampling the errors at a set σ of 160 equally spaced points on $[a,b]$. We denote these estimates by $e_M^{\{j\}}(n)$, i.e.

$$e_M^{\{j\}}(n) = \max_{x\in\sigma} \left| y^{(j)}(x) - \widetilde{Y}_M^{(j)}(x) \right| \quad ,$$

and in each table we also list the computed values

$$r_m^{\{j\}} = \log_2 \left\{ e_M^{\{j\}}(n) / e_M^{\{j\}}(2n) \right\},$$

giving the observed rates of convergence of $\widetilde{Y}_M^{(j)}$ to $y^{(j)}$.

Example 8.1 ([6: p.172])

$$\left.\begin{aligned} & y^{(2)}(x) + \frac{16x}{1+4x^2}\, y^{(1)}(x) + \frac{8}{1+4x^2}\, y(x) = 0, \quad x \in [0,1], \\ & y(0) = 1, y(1) = 0.2. \end{aligned}\right\} \tag{8.1}$$

Exact solution: $y(x) = \dfrac{1}{1+4x^2}$.

Numerical results: The results corresponding to the use of the extrapolated collocation method of [6] and to the quintic spline method of Section 6 are listed respectively in Tables 8.la and 8.1b. □

Example 8.2 ([15 :p.210])

$$\left.\begin{aligned} & y^{(4)}(x) + xy(x) = -(8+7x+x^3)e^x, \qquad x \in [0,1] \\ & y(0) = y(1) = 0, \qquad y^{(1)}(0) = 1, \qquad y^{(1)}(1) = e. \end{aligned}\right\} \tag{8.2}$$

Exact solution: $y(x) = x(1\text{-}x)\,e^x$.

Numerical results: The results corresponding to the extrapolated collocation method of [6] and to the $0(h^6)$ method of Section 5.3 are listed respectively in Tables 8.2a and 8.2b. □

Example 8.3

$$\left.\begin{aligned} &y^{(4)}(x)+4y(x)=1, \qquad x\in[-1,1], \\ &y(-1)=y(1)=y^{(2)}(-1) \ = y^{(2)}(1)=0 \end{aligned}\right\} \tag{8.3}$$

Exact solution: y(x) = 0.25{1-2(sin|sinh||sinxsinhx + cos|cosh| cosxcoshx)/(cos2+cosh2)}.

Numerical results: The results corresponding to the use of the $0(h^6)$ method of Section 5.3 are listed in Table 8.3; see Remark 5.1.

□

Example 8.4 ([4: p.603], [13 : p.25])

$$\left.\begin{aligned} &y^{(z)}(x)=\exp(y(x)) \ , \ x\in[0,1], \\ &y(0)=y(1)=0. \end{aligned}\right\} \tag{8.4}$$

Exact solution: y(x) = 2 ℓ n{csec[0.5c(x-0.5)]} - ℓ n2, where c is the unique solution of the equation $c = \sqrt{2}\,\cos(0.25c)$.

Numerical results: The results corresponding to the use of the extrapolated collocation method of [6] and to the quintic spline method of Section 6 are listed respectively in Tables 8.4a and 8.4b. In both cases, the approximating spline $\tilde{s}$ was determined, as indicated in Remark 7.5, by using Newton's method with initial approximation $\tilde{s}_0 = 0$. The iteration was terminated when the coefficients of the B-spline representation of $\tilde{s}_{(k+1)}$ agreed with those of $\tilde{s}_{(k)}$ to sixteen decimal places. In both cases this required five iterations.

TABLE 8.1a

Example 8.1 - *Extrapolated collocation method of [6]; see Section 5.2*

	M=0	M=1	M=2
j = 0	8.48E-8 4.1	7.04E-8 4.1	6.76E-8 4.0
j = 1	1.18E-5 3.0	1.54E-6 4.0	9.16E-7 4.8
j = 2	8.00E-3 2.0	5.65E-4 3.4	9.42E-5 4.0
j = 3	3.01E-0 1.0	2.52E-1 2.4	3.72E-2 3.0

Top entries: *Values of* $e_M^{\{j\}}(64)$ *Bottom entries:* *values of* $r_M^{\{j\}}(64)$

Theoretical rates: $r_M^{\{j\}} = \min\{4 - j + M, 4\}$

TABLE 8.1b

Example 8.1 - *$0(h^6)$ quintio spline method; see Section 6.*

	M=0	M=1	M=2
j = 0	4.55E-10 6.1	4.36E-10 6.1	1.12E-10 6.1
j = 1	1.16E-8 5.3	5.34E-9 6.0	9.80E-10 6.1
j = 2	4.31E-6 4.1	7.49E-7 5.4	7.18E-8 5.8
j = 3	1.51E-3 3.1	2.12E-4 4.2	9.65E-5 4.8

Top entries: *Values of* $e_M^{\{j\}}(64)$ *Bottom entries:* *Values of* $r_M^{\{j\}}(64)$

Theoretical rates: $r_M^{\{j\}} = min\,\{6 - j + M, 6\}$

TABLE 8.2a

Example 8.2 *Extrapolated collocation method of [6]; see Section 5.2*

	M=0	M=1	M=2
j = 0	6.14E-11 4.0	6.14E-11 4.0	6.14E-11 4.0
j = 1	2.10E-10 4.1	1.97E-10 4.0	1.96E-10 4.0
j = 2	9.14E-9 4.0	2.21E-9 4.0	2.21E-8 4.0
j = 3	2.96E-6 3.0	4.57E-8 4.0	1.54E-8 4.0
j = 4	1.95E-3 2.0	2.30E-5 3.4	3.62E-7 4.0
j = 5	7.51E-1 1.0	1.03E-2 2.2	9.31E-5 3.3
j = 6	—— ——	1.85E0 1.1	2.87E-2 2.1

Top entries: Values of $e_M^{\{j\}}(64)$ *Bottom entries:* Values of $r_M^{\{j\}}(64)$

Theoretical rates: $r_M^{\{j\}} = min\ \{6 - j + M, 4\}$

TABLE 8.2b

Example 8.2 - 0(h^6) quintic spline method; see Section 5.3

	M = 0	M = 1	M = 2	M = 3
j = 0	7.55E-12 6.0	3.47E-12 6.1	3.36E-12 6.1	3.36E-12 6.1
j =1	5.84E-10 5.1	4.34E-11 6.0	2.40E-11 6.0	2.43E-11 6.0
j = 2	1.24E-7 4.0	5.55E-9 5.0	1.89E-10 6.0	1.02E-10 6.0
j = 3	2.36E-5 3.0	7.69E-7 4.0	2.32E-8 5.0	1.01E-9 6.3
j = 4	7.63E-3 2.0	1.85E-4 3.4	7.37E-6 4.4	2.57E-7 5.3
j = 5	1.48E-0 1.0	4.82E-2 2.2	1.44E-3 3.2	5.09E-5 4.1
j = 6	— —	3.66E0 1.1	1.39-1 2.1	5.72E-3 3.1

Top entries: Values of $e_M^{\{j\}}(32)$ *Bottom entries:* Values of $r_M^{\{j\}}(16)$

Theoretical rates: $r_M^{\{j\}} = min\ \{6 - j + M, 6\}$

TABLE 8.3

Example 8.3 - $O(h^6)$ *quintic spline method; see Remark 5.1 of Section 5.3*

	M = 0	M = 1	M = 2	M = 3
j = 0	1.63E-12 5.8	1.63E-12 5.8	1.63E-12 5.8	1.63E-12 5.8
j = 1	7.33E-12 5.2	3.80E-12 5.9	3.41E-12 5.9	3.43E-12 5.9
j = 2	1.57E-9 4.0	1.14E-10 5.0	6.06E-12 6.0	4.02E-12 6.0
j = 3	2.89E-7 3.0	1.56E-8 4.0	7.31E-10 4.9	2.39E-11 5.5
j = 4	9.62E-5 2.0	6.72E-6 3.0	2.94E-7 4.0	7.17E-9 5.0
j = 5	1.85E-2 1.0	1.07E-3 2.0	4.45E-5 3.0	1.07E-6 4.0
j = 6	—— ——	8.28E-2 1.0	3.81E-3 2.0	9.86E-5 3.0

Top entries: Values of $e_M^{\{j\}}(64)$ *Bottom entries:* Values of $r_M^{\{j\}}(32)$

Theoretical rates: $r_M^{\{j\}} = min\ \{6 - j + M, 6\}$

TABLE 8.4a

Example 8.4 - *Extrapolated collocation method of [6]; see Section 7, Remark 7.2*

	M=0	M=1	M=2
j = 0	1.84E-10 4.0	6.28E-11 4.0	6.27E-11 4.0
j = 1	3.96E-8 3.0	9.18E-10 4.0	3.16E-10 4.1
j = 2	2.47E-5 2.0	5.78E-7 3.0	1.30E-8 4.0
j = 3	9.45E-3 1.0	1.90E-4 1.9	5.11E-6 2.9
j = 4	—— ——	2.96E-2 0.9	1.24E-3 1.9

Top entries: Values of $e_M^{\{j\}}(64)$ *Bottom entries:* Values of $r_M^{\{j\}}(32)$

Theoretical rates: $r_M^{\{j\}} = min\,\{4 - j + M, 4\}$

TABLE 8.4b

Example 8.4 - *$0(h^6)$ quintic spline method; see Section 7, Remark 7.4*

	M=0	M=1	M=2
j = 0	2.84E-14 5.8	2.85E-14 5.8	2.85E-14 5.8
j = 1	1.27E-12 5.1	1.72E-13 5.8	1.26E-13 5.8
j = 2	5.27E-10 4.0	2.29E-11 4.9	2.02E-12 5.8
j = 3	1.92E-10 3.0	6.24E-9 4.3	2.14E-9 4.8
j = 4	1.31E-4 2.0	6.01E-6 3.0	7.62E-7 3.9
j = 5	4.97E-2 1.0	1.84E-3 2.0	1.62E-4 2.9
j = 6	— —	2.80E-1 1.0	2.22E-2 1.9

Top entries: *Values of* $e_M^{\{j\}}(64)$ *Bottom entries:* *Values of* $r_M^{\{j\}}(32)$

Theoretical rates: $\dot{r}_M^{\{j\}} = \min\{6 - j + M, 6\}$

9. Discussion

We make the following three concluding remarks:

(i) The numerical results of Section 8 confirm the theory, and indicate that the methods of the present paper are capable of producing approximations of high accuracy. In particular, the results illustrate the substantial improvements in the accuracy of the approximations to $y^{(j)}$, $j > 0$, that can be achieved by the a posteriori correction of the approximating spline.

(ii) In the present paper we dealt only with the derivation and convergence theory of modified collocation methods. Thus, although the methods appear to be competitive, there is a clear need for a proper evaluation of their computational efficiencies. Such an evaluation will require a comparison analysis of the type carried out by Russell and Varah [14] and Russell [12], and will involve the study of computational aspects concerning, for example, the choice of representation for the approximating spline and the stability of the resulting matrix problems.

(iii) It will be of interest to investigate the possibility of extending some of the results f the present paper to partial differential equations. With reference to this, the methods of Archer [1] for quasilinear parabolic problems and of Houstis et al [7] for second order elliptic problems may be regarded as modified collocation methods. They correspond to the use of the corrected cubic spline approximation Y_1 of Remark 2.6(i), and can be considered to be extensions of the extrapolated collocation method of [6].

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